Entanglement in Theory Space

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We propose a new concept of entanglement for quantum systems: *entanglement in theory space*. This is defined by decomposing a theory into two by an un-gauging procedure. We provide two examples where this newly-introduced entanglement is closely related with conventional geometric entropies: deconstruction and AGT-type correspondence.

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Introduction.— Entanglement entropy is an indispensable measure for intrinsically quantum properties of quantum systems, and plays crucial roles in a number of different disciplines, such as quantum information and computation, many-body systems, quantum field theories and black hole physics (see e.g. Refs. [1–3] for reviews).

The goal of this Letter is to introduce a new concept of entanglement. As we will review momentarily the conventional definition of entanglement entropy involves the division of a spatial region into two. By contrast our entanglement entropy is defined by decomposing a gauge theory into two by an un-gauging of part of the gauge symmetry. Since the decomposition here involves not the geometric spatial regions but more abstract "space of quantum theories", our entanglement entropy will be called *entanglement in theory space*, or *theory-space entanglement*; for definiteness the conventional concept of entanglement will be hereafter called *geometric entanglement*.

While the concept of theory-space entanglement is rather unexplored and deserves further study, we point out that there are some examples where theory-space entanglement entropy is closely related with, or even equal to, the geometric entanglement entropy.

We believe that theory-space entanglement will provide useful tools to explore the *space of quantum field theories* in various dimensions, and learn about their mutual relations, perhaps along the lines of the Zamolodchikov metric for CFT.

Geometric Entanglement.— Let us first briefly recall the more conventional version, the geometric entanglement entropy.

Suppose that we have a quantum mechanical system; this could either be a discrete lattice system or a continuous field theory. In the canonical quantization we obtain a Hilbert space $\mathcal{H}_{\rm tot}$ on a time slice. Let us divide the spatial regions into a region A and its complement B; they share the boundary $\partial A = \partial B$. The total Hilbert space then factorizes into the product of those associated with regions A and B:

$$\mathcal{H}_{\rm tot} = \mathcal{H}_A \otimes \mathcal{H}_B \ . \quad (1)$$

Now consider the ground system of the system and the associated density matrix $\rho_{\rm tot}$. We can then define the reduced density matrix by

$$\rho_A = {\rm Tr}_{\mathcal{H}_B} \rho_{\rm tot} \ , \quad (2)$$

and the entanglement entropy $S_{\rm ent}$ as the von Neumann entropy for ρ_A: [23]

$$S_{\rm ent} = -{\rm Tr}_{\mathcal{H}_A} \rho_A \log \rho_A \ . \quad (3)$$

(Un-)Gauging.— In the definition above of the geometric entanglement entropy, the essential ingredients are that (i) there is a Hilbert space $\mathcal{H}_{\rm tot}$ and the ground state density matrix $\rho_{\rm tot}$ (ii) the total Hilbert space factorizes as in (1). We can then define the entanglement entropy $S_{\rm ent}$ by (2), (3).

While the spatial division gives rise to natural decomposition of the Hilbert space, it is not the only possibility. For example, Ref. [4] proposes entanglement entropy in the momentum space. Our proposal in this Letter is more drastic, and relies on the gauging/un-gauging procedure, which we now explain.

Suppose that we have two theories $\mathcal{T}_A$ and $\mathcal{T}_B$, with global symmetries G_A and G_B, respectively. In practice this means that the theory $\mathcal{T}_{A,B}$ is weakly gauged, i.e., has a coupling to the background gauge field for the global symmetry $G_{A,B}$. [24]

Suppose that $G_{A,B}$ contains a common subgroup G. We can then define a new theory $\mathcal{T}_{\rm tot}$ by identifying the corresponding background gauge fields and making them dynamical. In other words, we gauge the diagonal G-symmetry inside $G_A \times G_B$ (Fig. 4): [25]

$$\mathcal{T}_{\rm tot} = \mathcal{T}_A \cup_G \mathcal{T}_B \ . \quad (4)$$

Conversely we can decompose $\mathcal{T}_{\rm tot}$ into $\mathcal{T}_A$ and $\mathcal{T}_B$ by "un-gauging" the gauge symmetry G of $\mathcal{T}_{\rm tot}$. In practice this is to eliminate the kinetic term for the gauge field for G. When $\mathcal{T}_{A,B}$ have supersymmetry we can supersymmetrize the gauging procedure (4) and we add superpartners for the gauge fields.

We can regard the gauging as a procedure of constructing more complicated theories out of simple ingredients, and by repeating this procedure we obtain a zoo of quantum (field) theories with rich structures. In fact the idea of gauging has been extensively used, for example in the standard model of particle physics, and more recently in supersymmetric gauge theories. We will discuss some of these examples later.

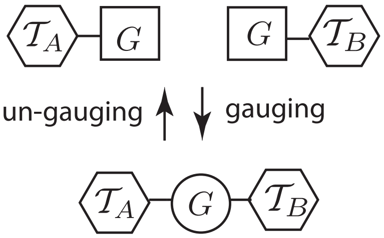

FIG. 1: Graphical representation of gauging/un-gauging. Here hexagons represent the theories, squares global symmetry, and a circle a gauge symmetry.

Theory-Space Entanglement.— We can now define the theory-space entanglement.

Suppose that $\mathcal{T}$ is a D-dimensional theory obtained by gauging two D-dimensional theories $\mathcal{T}_A$ and $\mathcal{T}_B$, as in (4). Let us consider the theory $\mathcal{T}$ on a D-dimensional manifold of the form $\mathbb{R}_t \times \mathcal{S}$, where $\mathcal{S}$ is a compact $(D-1)$-dimensional manifold and $\mathbb{R}_t$ is the time direction [26]. In the canonical quantization we obtain a Hilbert space $\mathcal{H}_{\rm tot}$ for a fixed time t, and the ground state density matrix $\rho_{\rm tot}$. By repeating this procedure for $\mathcal{T}_{A,B}$ we also obtain $\mathcal{H}_{A,B}$.

The basic idea is now clear: since we have $\mathcal{H}_{\rm tot}$, $\mathcal{H}_{A,B}$ and $\rho_{\rm tot}$, we can define the entanglement entropy by using the same formulas (2), (3).

There is one important subtlety, however; the factorization of the Hilbert space (1) does *not* hold, and we only have an embedding

$$\iota : \mathcal{H} \hookrightarrow \mathcal{H}_A \times \mathcal{H}_B \ . \tag{5}$$

The reason is that the states $|\psi_{A,B}\rangle$ in $\mathcal{H}_{A,B}$ in general is charged non-trivially under the global symmetry G, but then the product state $|\psi_A\rangle \otimes |\psi_B\rangle$ does not make sense as a state of $\mathcal{H}$ since it is not gauge invariant with respect to G, which is now promoted to a gauge symmetry in $\mathcal{T}_{\rm tot}$. Nevertheless we can defined the embedding (5) by incorporating the degrees of freedom for the gauge group G (and their superpartners) in the definition of $\mathcal{H}_{A,B}$, thus effectively doubling the degrees of freedom of G. To emphasize this some readers might prefer the notation $\mathcal{H}_{A+G}, \mathcal{H}_{B+G}$.

The non-factorization however is not really a problem, and a small modification saves the definition. The embedding ι induces the embedding of the ground state density matrix $\rho_{\rm tot} = |\psi_0\rangle\langle\psi_0|$:

$$\iota^*(\rho_{\rm tot}) := \iota(|\psi_0\rangle)\iota(\langle\psi_0|) \ . \tag{6}$$

Note by definition ι maps a pure state into a pure state. We modify the equation (2) by

$$\rho_A = {\rm Tr}_{\mathcal{H}_B} \iota^*(\rho_{\rm tot}) \ , \tag{7}$$

and can define the theory-space entanglement by the same equation (3). By the Schmidt decomposition it follows immediately that the answer does not change when we exchange the roles of A and B. This concludes our definition of theory-space entanglement. [27]

While the compactness of $\mathcal{S}$ removes IR divergences, there are in general still UV divergences, and the quantity defined above is divergent and regularization dependent; for a small UV regulator ϵ the answer diverges as powers of $1/\epsilon$. However, as in the case of geometric entropies, it is still possible that the constant (i.e. order ϵ^0) term or the coefficient of $\log \epsilon$ term is universal, depending on whether the dimension is odd or even.

The definition of the theory-space entanglement requires not the theory $\mathcal{T}$ itself, but also the choice of the compactification manifold $\mathcal{S}$ and the decomposition (4). Neither of these choices of unique. For example, since gauge symmetry is by definition a redundancy, theories with two different gauge groups could describe the same physics (prototypical example for this is the 4d Seiberg duality [5]), and the theory-space entanglement depends explicitly on the choice of the duality frame.

Comparison with Lattice Gauge Theories.— Some readers might be alarmed by the non-factorization of the Hilbert space (5), since standard treatment of entanglement entropy assumes factorization. However, let us point out that the factorization actually in general does not hold, even for conventional geometric entanglement entropies (see Refs. [6]).

The issue arises for gauge theories. For concreteness let us consider lattice gauge theories. In the Hamiltonian formulation the gauge-invariant degrees of freedom are given by strings of non-Abelian electric fluxes, and are not local in space [7]. Such strings in general spreads both in regions A and its complement B, and the spatial division violates the Gauss law on the boundary ∂A. This explains the non-factorization of the Hilbert space.

To put it another way, the problem is that in lattice gauge theories the basic degrees of freedom resides in the links connecting vertices, and not in the vertices. The boundary $\partial A = \partial B$ pass through some of the links, which are charged under some of the global symmetries. Note that this is not just a conceptual problem, but is of practical importance for numerical simulations of entanglement entropy.

To define geometric entropy for lattice gauge theories [6], we associate a new vertex for each link on the boundary and divide the link into two smaller links, one associated with region A and another region B. We then define the Hilbert space $\mathcal{H}_A$ ($\mathcal{H}_B$) to be the functionals of the connections of the links in region A (B) which are gauge invariant with respect to the gauge transformations associated with the vertices in the interior of A (B) but not necessarily with respect to the newly introduced vertices on the boundary. We then have the natural embedding (5) and the geometric entanglement entropy is defined by (7), (2). This is very analogous to the definition of the theory-space entanglement above.

The analogy goes even further in the context of decon-

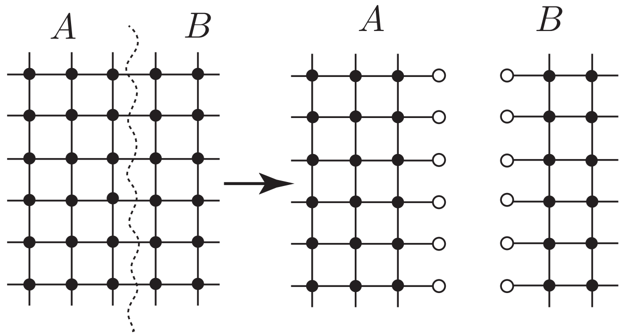

FIG. 2: In lattice gauge theories, the definition of geometric entanglement entropy requires the introduction of new vertices (colored white) and splitting of the link variables on the boundary.

struction [8, 9]. Let us begin with a quiver diagram on the circle, where quiver is simply a graph consisting of vertices and links (Fig. 3). Given a quiver we can construct a gauge theory by the rule that (1) we associate a $U(N_v)$ gauge group to each vertex v and (2) we associate a bifundamental matter with respect to $U(N_v) \times U(N_w)$ for a link connecting vertices v and w. The precise matter content can vary depending on the context, for example the amount of supersymmetry and the dimensionality of spacetime. For example (in the original example of Ref. [8]) we consider the four-dimensional theory, the link of the quiver is oriented, and the associated matter is a Weyl fermion with chirality determined by the orientation. For simplicity we take N_v to be independent of v, and denote the corresponding integer by N.

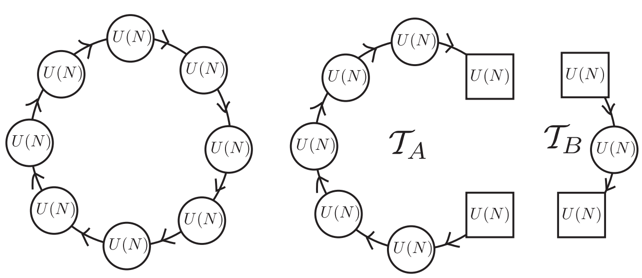

FIG. 3: The quiver diagram (left) for the deconstruction of an extra dimension. The circle direction along the quiver turns into a geometric extra dimension in the IR. The theory-space entanglement defined by dividing the quiver into two (right) corresponds to a geometric entropy of the deconstructed theory.

Now the claim of Ref. [8, 9] is that in the IR limit and in the limit of large number of quiver vertices, the 4d theory on the Higgs branch coincides with the 5d gauge theory, where only one of the directions is latticed the direction of the quiver becomes the extra dimension S^1_{extra} in the IR.

Since we have a 5d lattice gauge theory, we can define the geometric entanglement entropy along the S^1_{extra}, by dividing S^1_{extra} into two. More precisely let us take 5d lattice gauge theory dimensionally reduced on a compact 3-manifold $\mathcal{S}$, and consider the geometric entanglement for the resulting 2d theory on $\mathbb{R}_t \times S^1_{\text{extra}}$. [28]

As we have seen already, the definition of geometric entanglement in lattice gauge theories involves introducing new vertices on the boundaries of regions A, B. In the language of 4d quiver gauge theories, adding a node is translated into adding $U(N)$ symmetry. Since the Hilbert spaces $\mathcal{H}_{A,B}$ is not necessarily invariant under the $U(N)$ symmetry (as we discussed above), we should regard the $U(N)$ as a global symmetry acting on $\mathcal{H}_{A,B}$; to obtain $\mathcal{H}$ we need to gauge this symmetry. Since these are the same ingredients as in the definition of theory-space entanglement above [29], we learn that the geometric entanglement in the deconstructed 5d theory (dimensionally reduced on $\mathcal{S}$) coincides with the theory-space entanglement for the 4d quiver gauge theory (defined on the same manifold $\mathcal{S}$)! In other words we naturally arrive at the definition of the theory-space entanglement if we want to extend the notion of geometric entanglement of the deconstructed theory to the quiver gauge theory. This is one justification for our definition, and illustrates nicely the close relation between geometric and theory-space entanglement.

Geometric/Theory-Space Duality— Let us provide another (and more non-trivial) example of the relation between geometric and theory-space entanglement.

This examples deals with the cause of 4d $\mathcal{N} = 2$ superconformal field theories arising from the compactification of 6d $(2, 0)$ theories of type A_N on a punctured Riemann surface C [10]. From the viewpoint of 4d gauge theory, the geometry C is the defining data of the 4d theory.

A punctured Riemann surface C can be decomposed into a collections of three-punctured spheres (trinions) (Fig. 4). This is known as a pants decomposition. In the theories defined in Ref. [10], a trinion is associated with a theory called T_N, with global symmetries $SU(N)^3$; [30] each of the $SU(N)$ symmetries are associated with one of the punctures. When we glue such trinions, we gauge the diagonal of the associated $SU(N)$ global symmetries; this is the gauging procedure of (4). In other words, gauging of (4) for 4d gauge theories is translated into the geometrical operation of gluing $C = C_A \cup C_B$. Different choices of pants decompositions are argued to be different descriptions of the same 4d $\mathcal{N} = 2$ superconformal IR fixed point, and thus are S-dual to each other.

Since the definition of the theory involves a gauging (4), we can define the theory-space entanglement by compactifying the 4d theory on $\mathbb{R}_t \times \mathcal{S}$, where $\mathcal{S}$ is a compact 3-manifold. The Hilbert space $\mathcal{H}_{\mathcal{T}[C]}$ for our 4d theory $\mathcal{T}[C]$ depends on the choice of $\mathcal{S}$. We here choose a 1-parameter family of the 3-sphere S^3_b whose metric is given by $b^2(x_1^2 + x_2) + b^{-2}(x_1^2 + x_2) = 1$ [11].

Now the surprise is that the Hilbert space $\mathcal{H}_{\mathcal{T}[C]}$ contains a subspace ("BPS Hilbert space") $\mathcal{H}^{\text{BPS}}_{\mathcal{T}[C]}$ which co-

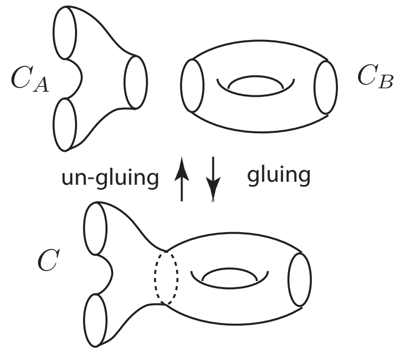

FIG. 4: The pants decomposition of the Riemann surface. In the geometric/theory-space duality of entanglement entropy, the gluing/un-gluing here is translated into the gauging/un-gauging of Fig. 1.

incides with the Hilbert space of the 3d $SL(N)$ Chern-Simons theory on the spatial Riemann surface:

$$\mathcal{H}[S_b^3]^{\text{4d BPS}}_{\mathcal{T}[C]} = \mathcal{H}^{\text{3d CS}}[\mathcal{C}] \ , \tag{8}$$

where the deformation parameter b of S_b^3 is translated into the level t of 3d $SL(N)$ Chern-Simons theory [12]. This is part of the statement of the "3d/3d duality" [12] (see also Refs. [13, 14]). In fact, we could regard S_b^4 of Refs. [15, 16] as S_b^3 fibered over an interval, with boundary conditions at both ends.

Let us consider the 6d $(2,0)$ theory on $\mathbb{R}_t \times S_b^3 \times C$. We can regard this either as (i) $(\mathbb{R}_t \times S_b^3) \times C$, giving rise to 4d $\mathcal{N}=2$ theory on $\mathbb{R}_t \times S_b^3$ or (ii) $(\mathbb{R}_t \times C) \times S_b^3$, giving rise to a 3d $SL(N)$ Chern-Simons theory on $\mathbb{R}_t \times C$ [12]. Since (8) is the equivalence of the Hilbert space, we automatically have the equivalence of the density matrix and the corresponding entanglement entropies. The correspondence is rather non-trivial since gauging of 4d $\mathcal{N}=2$ theories (Fig. 1) is translated into the geometrical gluing operation on the 2d surface (Fig. 4); the theory-space entanglement in the BPS Hilbert space of the 4d theory [31] is identified with the geometric entanglement in the 3d $SL(N)$ Chern-Simons theory on the geometric surface C!

We can also discuss a similar correspondence for a different compactification manifold $\mathcal{S}$; we can for example take the 4d theory on $S^1 \times S^3$, but with a twist along the S^1 direction [19]. The corresponding 3d theory is the $SU(N)$ Chern-Simons theory on $S^1 \times C$, which in turn gives 2d q-deformed Yang-Mills theory [17] on C (cf. Ref. [18]). [32]

Note in both of these cases the definition of entanglement entropy depends on the choice of the duality frame, and is not S-duality invariant.

Strong Subadditivity.— Geometric entropies satisfy one crucial relation, the strong subadditivity

$$\begin{aligned} &S_{\text{ent}}(A_1 \cup A_2) + S_{\text{ent}}(A_2 \cup A_3) \\ &\quad \geq S_{\text{ent}}(A_1 \cup A_2 \cup A_3) + S_{\text{ent}}(A_2) \ , \end{aligned} \tag{9}$$

for three spatial regions $A_{1,2,3}$.

We conjecture that there exists a counterpart for this statement in theory-space entanglement. To be concrete, suppose that the theory $\mathcal{T}$ has a decomposition into four:

$$\mathcal{T} = \mathcal{T}_1 \cup_{G_1} \mathcal{T}_2 \cup_{G_2} \mathcal{T}_3 \cup_{G_3} \mathcal{T}_4 \ . \tag{10}$$

It is then natural to define

$$\begin{aligned} &\mathcal{T}_{1\cup 2} = \mathcal{T}_1 \cup_{G_1} \mathcal{T}_2 \ , \mathcal{T}_{2\cup 3} = \mathcal{T}_2 \cup_{G_2} \mathcal{T}_3 \ , \\ &\mathcal{T}_{1\cup 2\cup 3} = \mathcal{T}_1 \cup_{G_1} \mathcal{T}_2 \cup_{G_1} \mathcal{T}_3 \ . \end{aligned} \tag{11}$$

The counterpart of (9) is

$$S_{\text{th}}(\mathcal{T}_{1\cup 2}) + S_{\text{th}}(\mathcal{T}_{2\cup 3}) \geq S_{\text{th}}(\mathcal{T}_{1\cup 2\cup 3}) + S_{\text{th}}(\mathcal{T}_2) \ , \tag{12}$$

where S_{th} is the theory-space entanglement defined with respect to the total theory $\mathcal{T}$ in (10). The proof of (12) will be similar to that of (9) (see e.g. Ref. [22]), however we have to carefully take the non-factorization (5) into account.

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[23] This also coincides with the von Neumann entropy for ρ_B.

[24] If we denote the current (background gauge field) for the global symmetry G by j_μ (A_μ), the Lagrangian contains the term $\mathcal{L} \supset \int A^\mu j_\mu$.

[25] After gauging the commutant of G inside $G_{A,B}$ remains as global symmetries of $\mathcal{T}_{A,B}$.

[26] It is not necessary to consider Lorentzian signature. For Euclidean signature the "time" is just one of the directions inside D-dimensions.

[27] We can generalize the definition to the case where we gauge the diagonal global symmetry for a set of theories $\mathcal{T}_{A_i}$, each with a global symmetry G. In the graphical representation of Fig. 1 this will be a multi-valent vertex.

[28] Alternatively we could choose to latticize all the four dimensions.

[29] The symmetry gauged in this case is $\prod_v U(N_v)$, where v runs over all the links on the boundary.

[30] Here we only consider the so-called full punctures. We can generalize the discussion to more general punctures labeled by Young diagrams.

[31] Due to boson-fermion cancellation the partition function on S_b^4 is the same for the BPS and the full Hilbert spaces [15]. However it is not clear if their theory-space entanglements are the same.

[32] It is tempting to speculate that similar correspondence holds for more general theories, such as those in Refs. [21], which discuss the relations between 4d superconformal indices and 2d spin chains.